IT Waves: Two Completed Generational Diffusion Models

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Information technology (IT) products and services, as well as many others, are released in successive generations that diffuse in wave-like patterns. The first generational model of such phenomena, the Norton-Bass Model, has been limited by conflicting views of what exactly it models, conflicting views on its properties, the belief by some that it has an inherent inconsistency and the fact that the model did not identify several desirable aspects of the modeled phenomena. This situation has developed largely because the Norton-Bass Model is actually two models in one, neither has been completely derived, and one of the models has never been rigorously defined and clearly differentiated from the other. This paper corrects these limitations. We define the quantities required in a "complete" generational diffusion model as the first step in completing the classic Norton-Bass as well as its newly defined Embedded Model. We estimate the newly defined model with two generations of wireless telephone subscribers, four generations of computer systems in use, two generations of copier sales and nine generations of DRAM sales. For all empirical cases we calculate sales, adopters, systems in use, switchers, leapfroggers, diffusion of the base technology, replacements, the cannibalization factor and other newly identified quantities.

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1.   Introduction

Information technology (IT) products and services, as well as many others, are released in successive

generations that diffuse in wave-like patterns such as in Figure 1. Generational shifts are of interest to

management in both IT-using organizations and supply-side companies (e.g., parts suppliers, end-use

product manufacturers, distribution companies). Interest in IT diffusion has expanded as managers of all

types are increasingly concerned about the generational transitions of their Web users (e.g., customers). It

is important to develop an understanding of the demand dynamics and interrelationships between product

generations. The advancement of such understanding is the purpose of this paper.

Figure 1. The Embedded Model Sales Fit to Gartner (2001) DRAM Data for Nine Generations
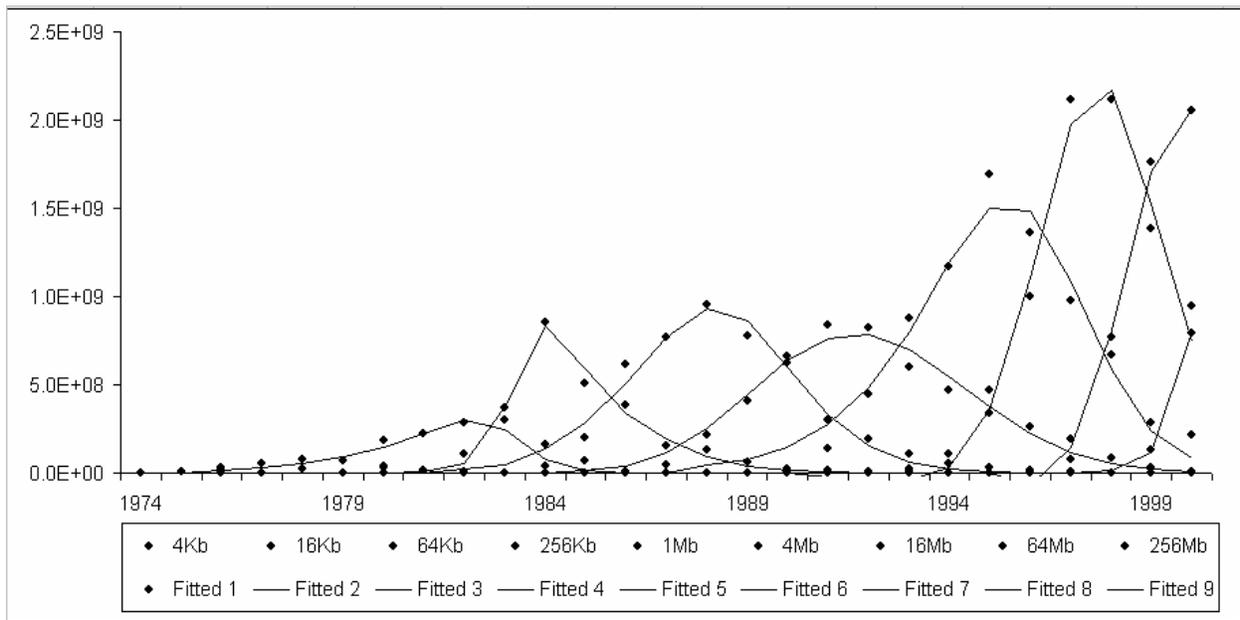
The first generational diffusion model was by Norton (1986) and Norton and Bass (1987). It is an

extension of the Bass Model (1963, 1969) of a single-generation product. Although the Norton-Bass has

proven useful, it has been limited by conflicting views of what exactly it models, conflicting views on its

properties, the belief by some that it has an inherent inconsistency and the fact that it did not identify sev-

eral desirable aspects of the modeled phenomena. This situation has developed largely because the Nor-

ton-Bass is actually two models in one, neither has been completely derived and one has never been rig-

orously defined and clearly differentiated from the other. This paper corrects these limitations.

Figure 2 shows the relationship of the two models that have been referred to as the Norton-Bass Model. We will call the one defined by Norton and Bass the "classic" Norton-Bass Model and denote it NB-87c. The "c" suggests the modeled phenomenon, continuous repeat purchasing by users. NB-87c was used by Speece and MacLachlan (1995). NB-87c was also discussed by Mahajan and Muller (1996) and Danaher et al. (2001) because the former said that it does not model systems in use and the latter implied that it does not model subscribers; these statements are true of NB-87c but not BB-04X, which does model these phenomena. We refer to the variant of the Norton-Bass that was described literarily, but not mathematically, by Norton and Bass and used by Islam and Mead (1997), Johnson and Bhatia (1997) as well as Kim et al. (2000) as the "Embedded" Model, BB-04X. We use "BB-04" because this paper is the first mathematical derivation of the model and we use "X" to suggest the modeled phenomenon, the count of users. We will later show that, as the names suggest, NB-87c and BB-04X are not the same model.

Figure 2. The Relationship of the Classic Norton-Bass (NB-87c) to its Embedded Model (BB-04X)
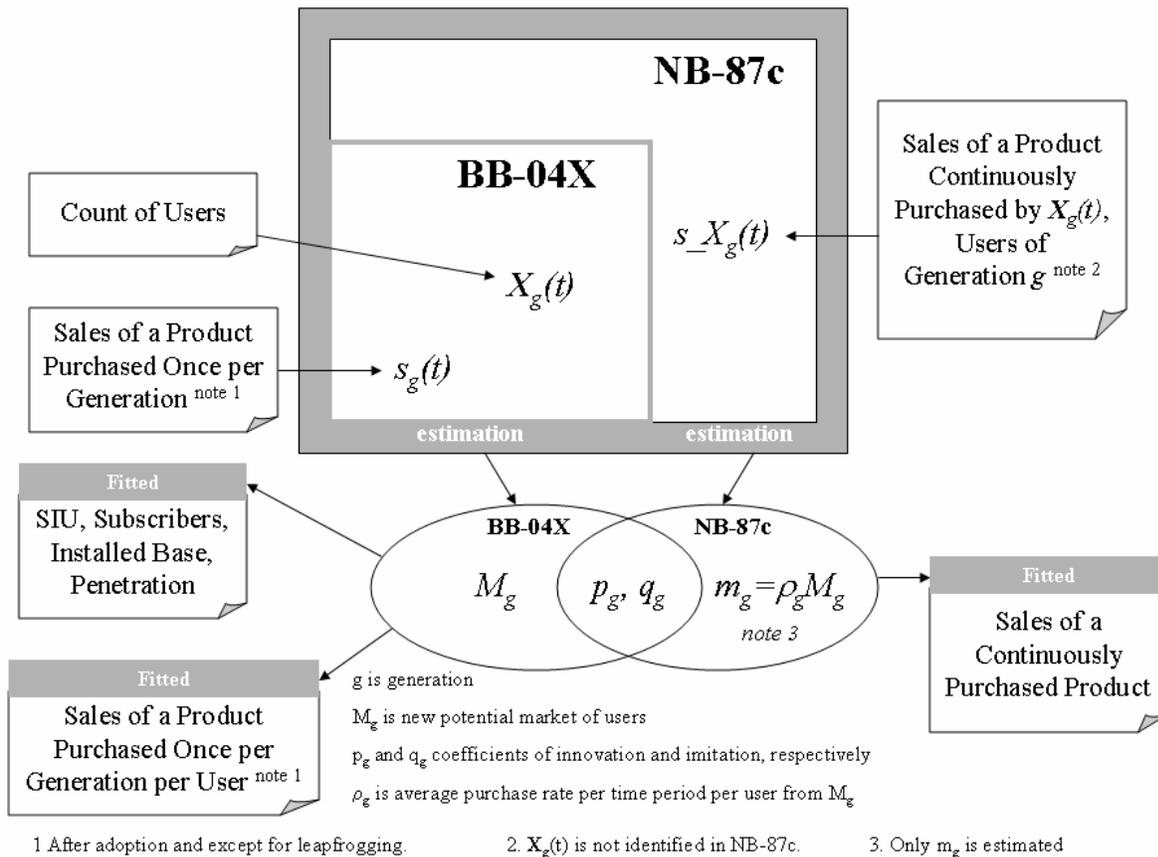
We define the quantities required in a "complete" generational diffusion model as the first step in completing the classic Norton-Bass NB-87c as well as its newly defined Embedded Model BB-04X. We develop BB-04X for G generations as two equations (one is recursive on g, the generation variable) and derive adoptions, sales, replacements, cannibalization, switching, leapfrogging and other quantities. BB-04X is more broadly applicable than NB-87c as the two equations that can be estimated are those most often required: count of users (e.g., subscribers, systems in use, installed base, penetration) and sales of a product purchased once each generation by each user except for leapfrogging. After we define BB-04X, we obtain NB-87c in the same manner as Norton and Bass did conceptually -- by assuming that each user makes an average number of repeat purchases each time period. The form of the equations in NB-87c is identical in BB-04X; however, all quantities have different meanings. This paper completes NB-87c by deriving sales due to users who leapfrog as well as other quantities that were heretofore not identified.

Finally, we estimate the BB-04X model of subscribers with two generations of wireless telephone subscribers and of systems in use (SIU) with four generations of IBM computers. We estimate the BB-04X derived model of sales with two copier generations and nine generations of DRAM shown in Figure 1, which we compare to the fit of NB-87c. For all cases we calculate switchers, leapfroggers, diffusion of the base technology, replacement purchases, the cannibalization factor and other quantities of interest.

The appendices referenced in this paper are available in Bass and Bass (2004).

2.   Literature Review – Norton and Bass

The product Norton and Bass had in mind was semiconductor chips, especially DRAM, which they viewed as continuously purchased by original-equipment manufacturers (OEMs) such as personal-computer (PC) companies, which incorporate chips into products bought by end users. The view reflects a common practice of forecasting chip sales as the number of customers and the average purchase rate. A new potential market was seen as coming in with each generation with timing based on the cumulative Bass (1969) distribution. The new and the old OEMs continuously purchase chips at an average rate per time period and switch from generation to generation. A similar method is used to estimate sales through

retail stores. Other products can be viewed as continuously purchased by consumers (e.g., diapers, music media). NB-87c is suitable, therefore, for any generational product that can be viewed as continuously purchased by a distribution channel (e.g., OEMs, stores) or consumers.

Norton (1986) [1] defined the gth generation sales of a continuously purchased product as

$$s\_X_g(t) = F_g(t)\Big[m_g + F_{g-1}(t)\Big[m_{g-1} + F_{g-2}(t)\Big[m_{g-2} + ...F_2(t)\big[m_2 + F_1(t)m_1\big]...\Big]\Big]\Big]\big[1 - F_{g+1}(t)\big], (1)$$

where, $F_g(t)$ is the cumulative Bass distribution, which for generation g is

$$F_g(t) = \left\{ \begin{array}{l} \dfrac{1 - e^{-(p_g+q_g)(t-\tau_g+1)}}{\left(1 + \left(\dfrac{q_g}{p_g}\right)e^{-(p_g+q_g)(t-\tau_g+1)}\right)^2}, t \geq \tau_g \\[3em] 0, \qquad\qquad t < \tau_g \end{array} \right\}, \qquad (2)$$

$\tau_g \geq 1$ is the known first year of generation g product shipments and p_g and q_g the parameters to be esti-mated. In the reduced model p and q do not have generational subscripts. For $1 \leq i < g \leq G,\ \tau_i \leq \tau_g$. m_g = $\rho_g M_g$ where M_g is the incremental potential market of generation g (applications, customers or sockets), and ρ_g is the average purchase rate of adopters originating in M_g. NB-87c does not identify ρ_g or M_g. We use $s\_X_g(t)$ rather than $S_g(t)$, used by Norton and Bass, for sales of a continuously purchased product. $s\_X_g(t)$ is not a cumulative quantity: it is sales at t to cumulative users of generation g. "$s\_X_g(t)$" does not mean s times $X_g(t)$, but rather that the name of the quantity is "s_X", suggesting "continuous product sales to cumulative users of generation g."

Norton and Bass (1987) estimated NB-87c with quarterly sales of DRAM (4 generations), SRAM (3 generations) and 8-bit logic (microprocessors followed by microcontrollers). Norton and Bass (1992) added to this with annual data for the three previously used industrial products plus two more (disk drives and drill bits), three pharmaceuticals (antihypertensives, blockers/inhibiters and diuretics), two consumer

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[1] The typographical errors in Norton (1986) are corrected here. See Appendix A for 4-generation case.

products (diapers and recording media) and computer sales in both units and performance units. All these

products can be viewed as continuously purchased except computer sales, which is also not SIU.

We know of three other generational models in scholarly papers since 1987. These and others have

suggested shortcomings of the Norton-Bass Model. Before discussing these papers, we will first review

Norton and Bass (1987, 1992) on two issues that have emerged: the p-q issue and the leapfrogging issue.

2.1. The p-q Issue

Which variant of NB-87c is best: (1) NB-87c-g with generation-specific Bass parameters p_g and q_g, or (2)

NB-87c-1, the reduced model with one p and one q? Norton and Bass (1987) specified NB-87c-g; but did

not report estimation. They argued for NB-87c-1 because: (1) customer-behavioral processes across gen-

erations are likely to be similar, (2) fits of NB-87c-1 were unlikely to be falsifiable (R^2 over .96 in one

case and over .98 in two) and (3) the forecasting advantages of NB-87c-1 are substantial because a new

generation can be forecasted prior to or soon after its launch by guesstimating only m_g. They reported

that only one NB-87c-g case converged and it had different generational p_g's and q_g's .

2.2. The Leapfrogging Issue

According to some, in the Norton-Bass Model users may not skip a generation (leapfrog). Norton and

Bass (1987) stated that users of a generation may have switched from an earlier one or have skipped the

earlier one to buy the later; in other words, users may switch or leapfrog although Norton and Bass did

not use those terms. They added that leapfroggers and switchers may or may not be distinguishable leav-

ing open the possibility of the mathematical relationships derived in this paper.

3. Additional Literature Review

In this section we review chronologically other research on generational diffusion models. Given the im-

portance of such patterns, especially for IT products, it is surprising we found only six published papers.

We review positions and methodologies as well as raise issues when we think it important for the ad-

vancement of knowledge in generational diffusion model research and application. We focus only on the

issues related to this paper; however, some papers have made other contributions, for example, a look at

optimal generational timing, the incorporation of price and the superimposition of intercategory effects. Although there are other multiproduct interactions (Bayus et al. 2000), we focus only on generations.

3.1. Speece and MacLachlan

Speece and MacLachlan (1995) applied NB-87c to three generations (e.g., paper) of gallon milk containers with a reported good fit. A separate poor fit to half-gallon containers caused them to modify NB-87c to include price and growth in a varying potential market. They did not explore including all substitutable products by combining gallons, half-gallons and other sizes into gallons of milk sold by generation of packaging (USDA 2002). They state "Implicit in the Norton and Bass Model is the assumption that technological substitution proceeds one generation at a time, that is, buyers do not immediately skip the second generation and jump to the third." In other words, NB-87c does not have leapfrogging.

3.2. Mahajan and Muller

Mahajan and Muller (1996) developed a generational model (MM-96) of SIU based on the Bass (1969) model because they viewed the Norton-Bass as applicable only to a continuously purchased good. One can infer they also viewed it as not identifying the diffusion of the base technology (category adoption). They defined leapfrogging as skipping to the last generation when switching from any previously purchased generation. MM-96 was estimated with data for four generations of IBM general-purpose computers, which as described in Bass (2002) left out sales of several third and fourth-generation IBM product families that were substantial successors to included first and second generation IBM computers. Mahajan and Muller used the model to explore optimal timing of new generations.

3.3. Islam and Meade

Islam and Meade (1997) claimed a "More General Model" because the Norton-Bass generational p-q variant gave a slightly better fit than the reduced form with the same p-q for all generations. Norton (1986) and Norton and Bass (1987, 1992) had previously defined and compared the same two models with different data. Islam and Meade did not describe the Norton-Bass Model as defined by the papers preceding them: Norton and Bass (1987, 1992), Mahajan and Muller (1996) and Speece and MacLachlan

(1995), instead they apparently unwittingly defined the Embedded Model BB-04X and fit it to subscriber and SIU data. They did indeed write about a more general model, but not the one they claimed.

Islam and Meade used quarterly cellular telephone data from 11 countries (2-3 generations) and the incomplete SIU data from Mahajan and Muller (1996). In the estimation with the cellular data, the difference in R^2 for BB-04X-g and BB-04X-1 averaged .002, which is very close. The data start in 1985; however, the first generation in some of the countries started earlier; for example, Sweden started in 1981. They did not mention adjusting the starting time to compensate for the missing data without which the estimated parameters would not be comparable to later generations. Forecasts would also be affected.

They concluded that the model with the greater number of parameters (BB-04X- g) is a better fit than the reduced model. They also concluded that forecasting is superior; but, they only explored extending the last generation after it is well underway, not forecasting prior to or just after a new generation starts. They argued that the total potential markets forecasted for the 11 cellular cases by BB-04X-g were superior because the forecast of BB-04X-1 was implausibly high. According to ITU (2003) data, history has reversed that judgment and shown the higher potential market in every case to be the better forecast.

Islam and Meade cite Mahajan and Muller (1996) as saying that the Norton-Bass Model has no leapfrogging; we found no such statement.

3.4.   Johnson and Bhatia

In the same year as Islam and Meade (1997), Johnson and Bhatia (1997) also used the Embedded Model BB-04X with three generations of mobile communications including land mobile radio, cellular and paging. They compared the resulting estimates of technological substitution to that obtained using best case regression techniques and reported superior results with BB-04X in all cases.

3.5.   Kim, Chang and Shocker

Kim et al. (2000) superimposed intercategory linkages on BB-04X to model effects of product categories on each other. They estimated the model with two cases: in Hong Kong one pager, two cellular telephone and one CT2 generation; and in Korea two pager and one cellular telephone generation.

3.6. Danaher, Hardie and Putsis

Danaher et al. (2001) developed a two-generation model (DHP-01) of a wireless telephone subscription service because they viewed the Norton-Bass Model as (1) not suitable for such phenomena -- even though Islam and Meade (1997), Johnson and Bhatia (1997) and Kim et al. (2000) had previously used it for wireless telephone --,;(2) as not having leapfrogging and (3) as having an unspecified "inherent inconsistency" when deriving the first-time purchase (adoption) model for multiple generations. Their model used a Bass (1969) model to construct new subscribers by generation with leapfrogging and switching constructed as functions of the cumulative distribution. Since the model is of only two generations there is no switcher leapfrogging, which requires at least three; therefore, their model does not have leapfrogging as defined by Mahajan and Muller (1996). Nor do they specify whether switcher leapfrogging would be incorporated if the model were extended to any number of generations or, if so, the form. Danaher et al. leapfrogging is by category adopters that originated in M_1 but purchased the second generation product first. The model was estimated with data of an unspecified European country (which we have determined is Sweden). They incorporate pricing and test three forms of the distribution function: the Bass, the Generalized Bass (Bass et al. 1994), and the Proportional Hazard.

Danaher et al. did not consider the effect of or estimate with the third generation, GSM, which was available in the Swedish market the last 25% of the time periods of the fitted data. In the last time period, GSM was 31% of subscribers (NPTAS 2000), which would likely have a substantial impact on prior generation parameter estimates as well as on the fit of a generational model.

3.7. Bass and Bass

We include Bass and Bass (2001) because the comparison of the model BB-01 to NB-87c spawned this research. BB-01 is a model of sales constructed from adoptions and replacement sales of a product purchased once each generation by each user except for leapfrogging. BB-01 has leapfrogging and features a derived equation for users (e.g., subscribers, systems in use) as an alternative equation to be estimated.

3.8.   Summary and Conclusions from Literature Review

The literature review raised fascinating questions about the desirable properties of, the proper use of and the selection of data for generational models, for example: (1) Need all significant substitutable and successive products be included in the data? (2) How should forecasting capability be judged? (3) Is leapfrogging a desirable property? (4) What is the effect of left truncating the data? (5) What is the effect of omitting the data of an existing generation in the timeframe? Answers to these are beyond our scope; but there are two even more fundamental questions for which we will provide some perspective. First, how should "generation" be defined? Second, what are the features of a "complete" generational model?

The literature review is consistent with the definition of a "generation" as triggered by a functionality and/or price improvement in a category so great that virtually all users will eventually switch to the new generation (unless an even newer one is introduced to which some leapfrog). An improvement sufficient to launch a new generation is not gradual: it is a step function. The improvement must be in the eyes of customers (Bass and Bass 2001, Bass and Bass 2004). Sometimes a manufacturer's claims of a "new generation," the popular view or the available data series do not pass this test.

A "complete" generational model must be defined for any number of generations and for any number of time periods. It enables calculation from estimated parameters all the quantities commonly of interest to diffusion modelers, sales forecasters and managers. Our literature review and our experience in modeling IT markets suggest the list in Table 1. The required quantities include counts of users by generation of product they are using (e.g., subscribers, SIU, installed base, penetration) as an equation to be estimated. Also required is the other commonly estimated equation, sales-1, which is for sales of a product purchased once each generation by each user except for leapfrogging (e.g., computers). The quantity Sales-c of a continuously purchased product is optional because there are large numbers of products for which it is not required. Also required are counts of users by generation of the market potential in which they originated and counts by generation of the product they first purchased (adopted). Total users over all generations is category diffusion. Other required quantities are switchers, renewals, replacements and the number of adoptions of each generation. We believe leapfroggers, over, from and to each generation

should be optional: we have no quarrel with the fact that leapfrogging occurs; however, the effect of this decomposed quantity, which has not been empirically validated, on a model needs more exploration.

Table 1. Complete Generational Diffusion Model – Required and Optional Quantities

Model Quantitiy	Symbol(s)	(Eq. Numbers) and Notes
Required -- In Embedded Model BB-04X		
Number of Generations	G	
Number of time periods	T	
Incremental market potential of generation g and total	M_g, M	
Count of users and change in count of users by generation of the product they are using (<u>Equations to be estimated</u>)	$X_g(t), x_g(t)$	(6), (10), (35) Subscribers, SIU, installed base, penetration
Sales-1 (new users) and Cumulative sales-1 by generation (<u>Equation to be estimated</u>)	$s_g(t), S_g(t)$	(32), (33) Sales of a product purchased once per generation except for leapfrogging
Total sales-1 and Total cumulative sales-1	$s(t), S(t)$	
Count of users and new users by generation of the product they first purchased (adopted)	$A_g(t), a_g(t)$	(34) Category adopters or just adopters
Count of users and new users by generation of the market potential in which they originated	$O_g(t), o_g(t)$	Below (6)
Total users and change in users summed by generation of the product being used, by the generation they adopted and by generation in which they originated	$X(t), x(t)$ -- $A(t), a(t)$ -- $O(t), o(t)$	(7), (35) Diffusion of the base technology (category diffusion)
Cumulative and new switchers from generation g	$W_g(t), w_g(t)$	(10) Also totals
Renewals of generation g product	$Z_g(t)$	(31)
Cumulative and new replacement sales by generation	$R_g(t), r_g(t)$	(36), (37) Repeats. Also totals
Optional -- In Embedded Model BB-04X		
New and cumulative leapfroggers over generation g	$y_g(t), Y_g(t)$	(12), (18) Cannibalized generation g sales. Also totals
New and cumulative leapfroggers to generation g	$yto_g(t), YTO_g(t)$	(22) Cannibalization by generation g Also totals
New and cumulative leapfroggers from generation g	$yfrom_g(t), YFROM_g(t)$	(25) Also totals
Leapfrogging adopters over, to and from generation g	$ya_g(t), yato_g(t), yafrom_g(t)$	(20), (23), (26) Cumulatives and totals also
leapfrogging switchers over, to and from generation g	$yw_g(t), ywto_g(t), ywfrom_g(t)$	(21), (24), (27) Cumulatives and totals also
The ultimate actual number of adoptions (first time category purchases) of generation g product and total	N_g, N	(38) Same as M_g except for leapfrogging
Optional -- In Classic Norton-Bass Model NB-87c		
Sales-c, Sales of a continuously purchased product to users of generation g (<u>Equation to be estimated</u>)	$s\_X_g(t)$	(1) This is Sales as defined by Norton and Bass

4. The Completion of the Classic Norton-Bass Model and its Embedded Model

In this section we define BB-04X, the Embedded Model, and then extend it to the classic NB-87c. We

first specify two generations to develop intuitive understanding of the model after which we define it for

any number of generations. This model is different from many others in that it has only a few key defined

quantities and relationships; however, many others can be derived from them, which we do.

Because this model has many variables it has been important to develop a consistent notation. All

quantities for both BB-04X and NB-87c are defined in Appendix C. Except for the Bass cumulative dis-

tribution $F_g(t)$, we use bold upper case for cumulatives with time and lower case for quantities that are not

cumulative. Characters without generational subscripts are the totals across all generations.

4.1. BB-04X (One and Two Generations)

We will refer to the product (or service) of generation g as P_g while a category P includes P_g for all g.

Writing the cumulative form of the Bass Model (1969) as a one-generation model, we have users of P_1

$$X_1(t) = M_1 F_1(t)$$
(3)

where the quantity M_g is the incremental potential market of adopters for generation g, t starting at 1 as

the introduction time of P_1 and $F_1(t)$ is the generational form of the cumulative Bass distribution, (2). M_g

is a factor of $m_g = \rho_g M_g$ of NB-87c, (1). In (3) once an adopter from M_1 actually adopts P_1, he "uses" it

indefinitely. $M_1 F_1(t)$ is users, cumulative adoptions, or cumulative sales; however, in multiple-generation

cases the three quantities – users, cumulative adoptions, and cumulative sales -- are different.

A second-generation product will bring into the market new potential adopters M_2. It will also cause

some potential adopters from M_1 to adopt the second generation instead of the first and will cause some of

the users of P_1 to switch to P_2. All purchases of P_2 regardless of any prior purchase history are governed

by the Bass distribution for the second generation. Building on (3) we have for two generations

$$X_1(t) = V_1(t) - V_1(t)F_2(t) \text{ where } V_1(t) = M_1 F_1(t)$$
(4)

$$\text{and } X_2(t) = V_2(t) \text{ where } V_2(t) = M_2 F_2(t) + V_1(t)F_2(t) .$$
(5)

In (4) $V_1(t)$ is potential users of P_1 while the subtracted quantity is those usurped by P_2 as allocated by

$F_2(t)$. Please understand that *$V_g(t)$* may or may not have ever used or ever will use P_g, but rather it is the

potential for being a user of P_g. The subtracted expression *$V_1(t)F_2(t)$* does not specify the way (switching

or leapfrogging) in which potential users of P_1 get to be potential users of P_2. What is subtracted in (4) is

added in (5) becoming part of the potential for being users of P_2. In (5) there is no future generation so

there is nothing subtracted. *$X_g(t)$* is the count of users of P_g at *t*: we do not call it "cumulative adopters."

The term "users" means cumulative count of users (as opposed to "change in" users) as do compara-

ble terms "systems in use," "installed base," "subscribers" and "penetration." We intentionally do not use

"cumulative adopters." "Adoptions" or "adopters" mean a first-time purchase. In the multiple-generation

case a purchase can be the first time any generation of the category is purchased or the first time a specific

generation is purchased; that is, category adoption or generation adoption. We use "adoption" to always

mean <u>category</u> adoption; thus, *$A_g(t)$* is the count of users of the category *P* at *t* by generation first pur-

chased. *$A_g(t)$* is not the same quantity as *$X_g(t)$*. *$X_g(t)$* is the count of users by generation they are using at

t. Some authors have referred to *$X_g(t)$* as cumulative adopters. We do not. Our distinction between *$A_g(t)$*

and *$X_g(t)$* is critical as we will derive both quantities.

4.2.   BB-04X (Any Number of Generations)

To extend (4) and (5) to *G* generations we define (cumulative) users of product generation *g* at *t* as

$$
X_g(t) = \left\{ \begin{array}{ll} V_g(t) - V_g(t)F_{g+1}(t), & g < G \\ V_g(t), & g = G \end{array} \right\}
$$
$$
\text{where } V_g(t) = \left\{ \begin{array}{ll} O_g(t), & g = 1 \\ O_g(t) + V_{g-1}(t)F_g(t), & 1 < g \leq G \end{array} \right\}
$$

(6)

where *$O_g(t) = M_g F_g(t)$* is cumulative "originating" potential users and *$X_g(t)$, $V_g(t)$, $F_g(t)$* and M_g are as de-

scribed in the preceding section. The Embedded Model BB-04X is completely defined in (6). *$V_g(t)$* is

recursively defined on *g*, which enables (6) to be more concise and easier to work with than prior rendi-

tions of the Norton-Bass; for example, (1). In later sections we will develop equations that provide addi-

tional insight and derive additional quantities (e.g., leapfroggers); but the derived quantities require no

additional constructions: they are already implied by (6) and need only be derived, recognized as the phe-

nomenon aspects they represent and named. First, however, we will look at some interesting properties of (6) and obtain NB-87c from BB-04X and vice versa as well as prove that NB-87c and BB-04X are not the same model.

4.3.   Two Important Properties of BB-04X

Appendix B explores (6) through algebraic manipulation that gives additional insight. Here we look at two properties that are especially of interest. If we look at the total over g of $X_g(t)$ from (6), we have

$$X(t) = \sum_{g=1}^{G} X_g(t) = \begin{bmatrix} \boldsymbol{O}_1(t) - \boldsymbol{U}_1(t) + \\ \boldsymbol{O}_2(t) + \boldsymbol{U}_1(t) - \boldsymbol{U}_2(t) + ... + \\ \boldsymbol{O}_{G-1}(t) + \boldsymbol{U}_{g-2}(t) - \boldsymbol{U}_{G-1}(t) + \\ \boldsymbol{O}_G(t) + \boldsymbol{U}_{G-1}(t) \end{bmatrix} = \sum_{g=1}^{G} \boldsymbol{O}_g(t) = \boldsymbol{O}(t), \tag{7}$$

where the usurped quantity $\boldsymbol{U}_g(t)$ is

$$\boldsymbol{U}_g(t) = \left\{ \begin{array}{cc} \boldsymbol{V}_g(t) F_{g+1}(t), & g < G \\ 0, & g = G \end{array} \right\}. \tag{8}$$

The usurped quantities in (7) cancel leaving the total of users by generation of market potential M_g in which they originated on the right. Note that although $\boldsymbol{X}(t) = \boldsymbol{O}(t)$, $\boldsymbol{X}_g(t) \neq \boldsymbol{O}_g(t)$. Both $\boldsymbol{X}(t)$ and $\boldsymbol{O}(t)$ are the diffusion of the base technology (category diffusion) referred to by Mahajan and Muller (1996).

The fact that eventually all buyers of earlier generations will migrate to generation G is shown by

$$\lim_{t\to\infty} \boldsymbol{X}_g(t) \left\{ \begin{array}{ll} = \lim_{t\to\infty} \left[\boldsymbol{V}_g(t)\left(1 - F_{g+1}(t)\right) \right] = 0, & g < G \\ = \lim_{t\to\infty} \left[\boldsymbol{V}_g(t) \right] = \sum_{j=1}^{G} M_j, & g = G \end{array} \right\}, \tag{9}$$

which as shown in Appendix D, follows by using $\boldsymbol{V}_g(t)$ from (6) and the fact that $F_g(\infty)=1$.

4.4.   Obtaining the Norton-Bass (NB-87c) from its Embedded Model (BB-04X) and Vice Versa

To obtain NB-87c from BB-04X we add the assumption that every adopter in M_g purchases an average of ρ_g products each time period that he/she/it is a user. We modify (6) by changing M_g to m_g where $m_g=\rho_g M_g$. As a result, $\boldsymbol{X}_g(t)$ and $\boldsymbol{V}_g(t)$ change to s_X_g (sales to users of P_g) and s_V_g (sales to potential

users of P_g), respectively. Equation (6) of BB-04X is exactly the same functional form as the corresponding equation of NB-87c (see Appendix A) but all terms have different meanings. This is similar to an example in physics (Gullberg 1997) where the same functional forms and relationships with different meanings describe two different phenomena: mechanical and electrical.

Alternatively, BB-04X can be obtained from NB-87c. One might think that because of the similarity of mathematical form in conjunction with the relationship $m_g = \rho_g M_g$, to obtain the model BB-04X from NB-87c one need only set $\rho_g = 1$ thus effectively replacing m_g with M_g. This is not so although the resulting functional form is the one sought. The model thus obtained is one in which each user buys <u>one new product each time period</u> making the limit over time of the last generation be

$$\lim_{t \to \infty}\left[\sum_{i=1}^{t}[s\_X_G(t)]\right] = \lim_{t \to \infty}\sum_{i=1}^{t}\left[\lim_{t \to \infty}[s\_X_G(t)]\right] = \lim_{t \to \infty}\sum_{i=1}^{t}\left[\sum_{g=1}^{G}m_g\right] = \lim_{t \to \infty}\sum_{i=1}^{t}\left[\sum_{g=1}^{G}\rho_g M_g\right] = \lim_{t \to \infty}\sum_{i=1}^{t}\left[\sum_{g=1}^{G}M_j\right] = \infty,$$

which uses the result from (9) restated in NB-87c symbols as the third step. In (9) the limit is not infinity; therefore, the model obtained by just setting $\rho_g = 1$ is not BB-04X. To obtain BB-04X we must change the NB-87c assumption that each user buys an average number of products each time period to the assumption that each user buys one product each generation (except for leapfrogging) and uses it until switching. The new assumption is counting users; therefore, the new definitions of $X_g(t)$ and $V_g(t)$ are "users of P_g" and "potential users of P_g," respectively. The practical importance of this is that NB-87c is fit to sales of a continuously purchased product; while BB-04X, (6), is fit to systems in use, installed base, subscribers or penetration. Some have thought BB-04X is a special case of NB-87c but given the required change in definitions this is not so. Appendix C gives definitions of all variables in both models.

Although NB-87c can be obtained from BB-04X and vice versa by substituting one assumption for another and changing the variable meanings, NB-87c and BB-04X are not the same model. If we consider them the same, we have a contradiction as the same equation should be fit to both sales and SIU. Further, because we will derive from (6) sales of a product that is *not* continuously purchased, we would

have two definitions of sales in the same model. Some have viewed this as a problem with the model

when actually it is the fact that there are two different models albeit one is embedded in the other.

4.5.   Derivation of BB-04X Leapfroggers and Switchers

Derivations of many quantities (e.g., sales) in BB-04X depend on separately identifying leapfroggers and

switchers. To do so requires that we work with the discrete form; therefore, we will use the "change in"

(also referred to as "delta") portion of the cumulative Bass distribution as (Srinvasan and Mason 1986).

We will refer to a delta quantity as "new" only if there are no negative quantities in the change or if we

mean only the positive part of the change.

As shown in Appendix B, differencing (6) is straightforward and yields the delta quantities:

$$
\begin{aligned}
x_g(t) &= \begin{cases} v_g(t) - u_g(t), & g < G \\ v_g(t), & g = G \end{cases}, \\
v_g(t) &= \begin{cases} o_g(t), & g = 1 \\ o_g(t) + u_{g-1}(t), & 1 < g \le G \end{cases}, \\
o_g(t) &= M_g f_g(t).
\end{aligned}
\tag{10}
$$

$x_g(t)$, $v_g(t)$ and $o_g(t)$ in (10) are expressed as simple functions of other delta quantities and M_g. The

quantity that we will now explore, $u_g(t)$, is more complex and in its complexity we will discover and

prove the identities of leapfroggers and switchers. We are not surprised that $u_g(t)$ is more complex be-

cause we have understood from the definition of $U_g(t)$ in the Norton-Bass Model that it contains two types

of usurped potential users: those who skipped P_g and those who owned P_g. Nor are we surprised that the

discrete form is required to separately identify leapfroggers and switchers because both concepts, "leap-

froggers" and "switchers," depend on being able to write an expression involving the state of P_g owner-

ship for a portion of users in the time period before the current one: a concept that exists only in a discrete

model. We write delta usurped potential users $u_g(t)$ as the difference equation $U_g(t)$-$U_g(t-1)$ using (8).

Substituting $V_g(t-1)+v_g(t)$ for $V_g(t)$ then $f_g(t)$ for $F_g(t)$-$F_g(t-1)$ for $t>1$ yields

$$u_g(t) = \begin{cases} \begin{cases} \begin{aligned} &U_g(t) - U_g(t-1) \\ &= V_g(t)F_{g+1}(t) - V_g(t-1)F_{g+1}(t-1) \\ &= \left[V_g(t-1) + v_g(t)\right]F_{g+1}(t) - V_g(t-1)F_{g+1}(t-1) \\ &= \left[V_g(t-1)F_{g+1}(t) - V_g(t-1)F_{g+1}(t-1)\right] + v_g(t)F_{g+1}(t) \\ &= V_g(t-1)\left[F_{g+1}(t) - F_{g+1}(t-1)\right] + v_g(t)F_{g+1}(t) \\ &= V_g(t-1)f_{g+1}(t) + v_g(t)F_{g+1}(t) \\ &= w_g(t) + y_g(t) \end{aligned} \end{cases}, \quad g < G \\ \quad 0, \hspace{6cm} g = G \end{cases} \tag{11}$$

where
$$y_g(t) = v_g(t)F_{g+1}(t), \tag{12}$$

which we will show is leapfroggers over P_g at t, and

$$w_g(t) = V_g(t-1)f_{g+1}(t), \tag{13}$$

which we will show is switchers from P_g at t. Since we want to explore P_g ownership at t-1 by $w_g(t)$ and $y_g(t)$, we write users at t as users at t-1 plus new users at t using (10) and (11) for $g < G$ as

$$\begin{cases} X_g(t) = X_g(t-1) + x_g(t) \\ \quad = X_g(t-1) + v_g(t) - w_g(t) - y_g(t) \end{cases}. \tag{14}$$

In (14) we see that $X_g(t)$, which are P_g users at t, consists of $X_g(t$-$1)$, which were P_g users at t-1, plus $v_g(t)$, which is the new potential users, minus the quantities $w_g(t)$ and $y_g(t)$. We will show that $w_g(t)$ were P_g users at t-1 and that $y_g(t)$ were not; thus, they were switchers and leapfroggers, respectively.

Since the first term is the same, we will show that $w_g(t) = V_g(t-1)f_{g+1}(t)$ from (13) is part of $X_g(t$-$1)$, (6), which is

$$X_g(t-1) = V_g(t-1)\left[1 - F_{g+1}(t-1)\right], \tag{15}$$

by showing that $f_{g+1}(t)$, (13), is part of $[1-F_{g+1}(t$-$1)]$, a factor of (15) and the complement of $F_{g+1}(t$-$1)$. By definition, $f_{g+1}(t) = F_{g+1}(t) - F_{g+1}(t$-$1)$; so $f_{g+1}(t)$ is not part of $F_{g+1}(t$-$1)$ and must be part of its complement, $[1-F_{g+1}(t$-$1)]$. Therefore, $w_g(t)$ is part of $X_g(t$-$1)$ and were P_g users at t-1 and are indeed switchers. From (12), $y_g(t) = v_g(t)F_{g+1}(t)$, is <u>not</u> part of $X_g(t$-$1)$, (15), because by definition, $v_g(t) = V_g(t) - V_g(t$-$1)$; thus

$v_g(t)$ is not part of $V_g(t-1)$; so $y_g(t)$ were not P_g users at $t-1$ and are indeed leapfroggers over P_g. We observe from (12) and (13) that $y_g(t)$ and $w_g(t)$ are distinct because $v_g(t)$ is not part of $V_g(t-1)$. $y_g(t)$ and $w_g(t)$ are nonzero because from (6) we can conclude that the factors are non-zero if any $M_j \neq 0$ for $j \leq g$.

4.6. The Interaction of Leapfrogging and Switching

We can now explore the interaction between switching and leapfrogging and show that as many as *G-2* generations can be leapfrogged. Using (6) and (8) to substitute for $V_g(t-1)$ in (13) we have for $1 \leq g < G$

$$w_g(t) = \left[\mathbf{O}_g(t-1) + \mathbf{U}_{g-1}(t-1) \right] f_{g+1}(t) \; ; \tag{16}$$

then substituting $\mathbf{U}_{g-1}(t-1) = \mathbf{W}_{g-1}(t-1) + \mathbf{Y}_{g-1}(t-1)$, we have those who switch away from P_g at t as

$$w_g(t) = \begin{cases} \left[\mathbf{O}_g(t-1) + \left[\mathbf{W}_{g-1}(t-1) + \mathbf{Y}_{g-1}(t-1) \right] \right] f_{g+1}(t), & 1 \leq g < G \\ 0, & g = G \end{cases} \tag{17}$$

we see that those who switch away from P_g at t includes: (1) those from M_g potential who bought P_g as their first product, (2) those who bought P_g after switching away from P_{g-1} and (3) those that bought P_g after leapfrogging P_{g-1}. Using (12) substitute for $v_g(t)$ from (10) then for $u_{g-1}(t)$ from (11) to obtain

$$y_g(t) = \begin{cases} o_g(t)F_{g+1}(t), & g = 1 \\ \left[o_g(t) + w_{g-1}(t) + y_{g-1}(t) \right] F_{g+1}(t), & 1 < g < G \\ 0, & g = G \end{cases} , \tag{18}$$

which shows those who leapfrog P_g include: (1) those from M_g who do not adopt P_g, (2) those who skip P_g as they switch away from P_{g-1} and (3) those who skip P_g as they skip P_{g-1}. The first type of leapfrogging is leapfrogging adoption, the second is leapfrogging switching and the third type can be either. The recursive definition of $y_g(t)$ on g makes clear that many generations can be leapfrogged. From (16)-(18) we see that many combinations of leapfrogging and switching are possible; therefore some customers from M_g buy every product generation starting with g while some will leapfrog to buy P_{g+1} first, and so on to some will buy P_G first. Others from M_g will do some combination of switching and leapfrogging.

4.7. Leapfrogging Adopters, Leapfrogging Switchers and Leapfroggers Over, From and To P_g

We can separate leapfrogging adopters from leapfrogging switchers by writing (18) as
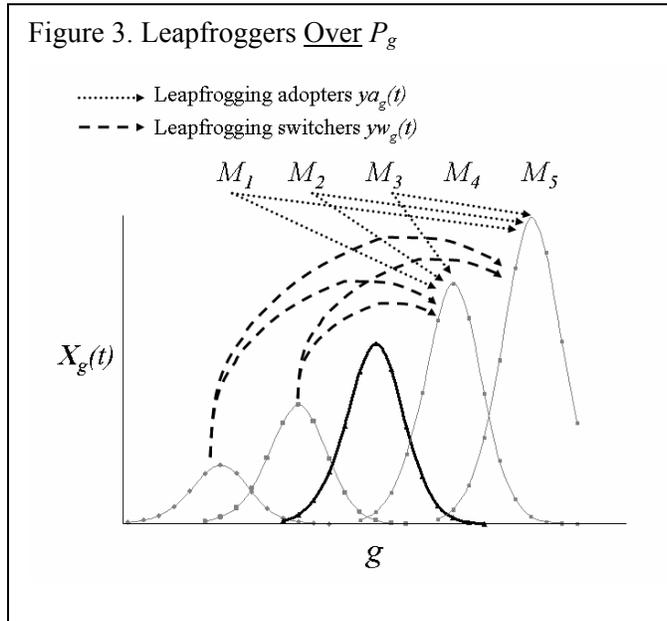
$$y_g(t) = ya_g(t) + yw_g(t) \tag{19}$$

where leapfrogging adopters over P_g are

$$ya_g(t) = \begin{cases} o_g(t)F_{g+1}(t), & g = 1 \\ \left[o_g(t) + ya_{g-1}(t)\right]F_{g+1}(t), & 1 < g < G \\ 0, & g = G \end{cases} \tag{20}$$

and leapfrogging switchers over P_g are

$$yw_g(t) = \begin{cases} 0, & g = 1 \\ \left[w_{g-1}(t) + yw_{g-1}(t)\right]F_{g+1}(t), & 1 < g < G \\ 0, & g = G \end{cases}. \tag{21}$$

Figure 3. Leapfroggers Over P_g

········▶ Leapfrogging adopters $ya_g(t)$

– – –▶ Leapfrogging switchers $yw_g(t)$

M_1 M_2 M_3 M_4 M_5

$X_g(t)$

g

Leapfrogging adopters, $ya_g(t)$, are adopters from M_g and prior that skip P_g. Leapfrogging switchers, $yw_g(t)$, are P_{g-1} or prior users who skip P_g when switching to a later generation. Both leapfrogging adopters and leapfrogging switchers skip over P_g as illustrated in Figure 3. We will also develop leapfroggers who skip from P_g and separately leapfroggers who skip to P_g. Leapfroggers over a generation are "cannibalized." Leapfroggers to P_g are

$$yto_g(t) = yato_g(t) + ywto_g(t) \tag{22}$$

where leapfrogging adopters to P_g are leapfrogging adopters that skip P_{g-1} minus those that skip on,

$$yato_g(t) = \begin{cases} 0, & g = 1 \\ ya_{g-1}(t) - ya_{g-1}(t)F_{g+1}(t), & 1 < g < G \\ ya_{g-1}(t), & g = G \end{cases}, \tag{23}$$

and leapfrogging switchers to P_g are those that skip P_{g-1} while switching and did not skip on,

$$ywto_g(t) = \begin{cases} 0, & g = 1 \\ yw_{g-1}(t) - yw_{g-1}(t)F_{g+1}(t), & 1 < g < G \\ yw_{g-1}(t) & g = G \end{cases}. \tag{24}$$

The quantity $yto_g(t)$ is sales cannibalized by P_g. Leapfroggers <u>from</u> P_g are

$$yfrom_g(t) = yafrom_g(t) + ywfrom_g(t), \tag{25}$$

where leapfrogging adopters <u>from</u> M_g market potential are

$$yafrom_g(t) = \begin{cases} o_g(t)F_{g+1}(t), & g < G \\ 0, & g = G \end{cases} \tag{26}$$

and leapfrogging switchers <u>from</u> P_g are switchers from P_g that leapfrog to P_{g+2} (or later)

$$ywfrom_g(t) = \begin{cases} w_g(t)F_{g+2}(t), & g \le (G-2) \\ 0, & g > (G-2) \end{cases}. \tag{27}$$

By expanding the totals (see Appendix D), we can show that the following relationships hold:

$$yfrom(t) = yto(t), \, yafrom(t) = yato(t) \text{ and } ywfrom(t) = ywto(t). \tag{28}$$

Finally, we define the cannibalization factor $C_g(t)$ as the percentage of potential users of P_g, that instead leapfrog, the total cannibalization factor $C(t)$ as the percentage over all generations except the last for which leapfrogging is not possible, and C as the ultimate total cannibalization factor:

$$C_g(t) = \frac{Y_g(t)}{V_g(t)}, \quad C(t) = \frac{\sum_{g=1}^{G-1} Y_g(t)}{\sum_{g=1}^{G-1} V_g(t)} \text{ and } C = \lim_{t \to \infty} C(t). \tag{29}$$

4.8.   Derivation of Sales, Renewals, Adoptions, Replacements and Diffusion of the Base Technology

Now that leapfroggers and switchers have been separately identified, we can derive other quantities.

We can rewrite users from (14) as

$$\begin{aligned} X_g(t) &= \left[X_g(t-1) - w_g(t) \right] + \left[v_g(t) - y_g(t) \right] \\ &= \left[Z_g(t) \right] + \left[s_g(t) \right], \end{aligned} \tag{30}$$

where $Z_g(t)$ is old users who don't switch,

$$Z_g(t) = X_g(t-1) - w_g(t),$$ (31)

which we recognize as renewals (a cumulative quantity) as in Danaher et al. (2001) and $s_g(t)$ is new users,

$$s_g(t) = v_g(t) - y_g(t),$$ (32)

which we recognize as sales of P_g at t. Conceptually, sales is potential users who do not leapfrog. Notice that sales is not (10), the change in users. Substituting in (32) for $v_g(t) = x_g(t) + u_g(t)$ from (10) and for $y_g(t) = u_g(t) - w_g(t)$ from (11), sales is the change in users plus those who switch to a later generation

$$s_g(t) = v_g(t) - y_g(t) = \left[x_g(t) + u_g(t)\right] - \left[u_g(t) - w_g(t)\right] = x_g(t) + w_g(t).$$ (33)

Conceptually (33) recognizes that the switchers have been subtracted from change in users and it should not be subtracted from sales. Equations (32) and (33) are sales of a product purchased once each generation by each user except for leapfrogging such as sought by Bass and Bass (2001). These equations can be fit to data such as PC sales, copier sales and word processing software sales.

We can now use previously derived quantities to logically formulate new category adopters of P_g at t as those who originate in M_g minus those who leapfrog from M_g when adopting plus those who leapfrog to P_g when adopting

$$a_g(t) = o_g(t) - yafrom_g(t) + yato_g(t),$$ (34)

which with the cumulative $A_g(t)$ are category adopters as sought by Bass and Bass (2001). With (7) we now have three equal expressions (see Appendix D) for category diffusion (diffusion of the base technology). The three expressions are equal because they are three different ways of counting the same thing (total change in users at t): by the originating generation, by generation using and by generation adopted. The three expressions are: total new originating users, total change in users and total new adopters:

$$o(t) = x(t) = a(t), \text{ but } x_g(t) \neq o_g(t), x_g(t) \neq a_g(t) \text{ and } o_g(t) \neq a_g(t).$$ (35)

Sales that are not category adoptions are replacement sales as sought by Bass and Bass (2001):

$$r_g(t) = s_g(t) - a_g(t). \tag{36}$$

As shown in Appendix D, if we use (36) to substitute for $s_g(t)$ from (32) and for $a_g(t)$ from (34) then apply in order relationships in (26), (23), (10), (11), (20), (21) and (21) again we obtain

$$r_g(t) = w_{g-1}(t) + yw_{g-1}(t) - yw_g(t); \tag{37}$$

thus replacements is also new switchers to P_g from any prior generation ,which is switchers from P_{g-1} plus leapfrogging switchers over P_{g-1} less switchers that leapfrog P_g. Replacements would be simply switchers from the prior generation except for leapfrogging; that is, $r_g(t) \neq w_{g-1}(t)$; however, the totals $r(t) = w(t)$.

Recall that M_g is the new potential market with generation g. M_g adopters do not necessarily adopt P_g, however, because of leapfrogging when adopting; therefore, we introduce the quantity N_g, which we define as the ultimate number of actual adoptions of P_g,

$$N_g = \lim_{t \to \infty} A_g(t) = A_g(\infty). \tag{38}$$

If there is no leapfrogging, $yafrom_g(t) = 0$ and $yato_g(t) = 0$ for all t and using (34) and (38)

$$N_g = \lim_{t \to \infty} \sum_{i=1}^{t} a_g(i) = \lim_{t \to \infty} \sum_{i=1}^{t} \left[M_g f_g(t) - yafrom_g(t) + yato_g(t) \right] = \lim_{t \to \infty} \sum_{i=1}^{t} M_g f_g(t) = M_g; \tag{39}$$

therefore, N_g is M_g except for adoption leapfrogging. Of course, in all cases, the total $M = N$.

4.9.   Cumulatives with Time and Totals over all Generations

In the preceding sections many new quantities were derived as the "change in" at t of an unspecified corresponding cumulative ("count of") , for example, leapfroggers $y_g(t)$ of the cumulative $Y_g(t) = \sum_{i=1}^{t} y_g(i)$.

Other cumulatives are similarly defined. Totals across all generations of any non-cumulative quantity such as $y(t) = \sum_{g=1}^{G} y_g(t)$ and of any cumulative quantity such as $Y(t) = \sum_{g=1}^{G} Y_g(t)$.

4.10.   Derivation of New Quantities for NB-87c

To extend BB-04X to NB-87c we need only remember that any quantity in the Embedded Model such as $y_g(t)$ leapfroggers of generation g at t in NB-87c is $s\_y_g(t)$ sales due to leapfroggers of generation g at t. A

cumulative quantity in BB-04X such as cumulative users $X_g(t)$ in NB-87c is not cumulative: it is $s\_X_g(t)$, sales at t due to cumulative users. To derive NB-87c expressions, M_g in BB-04X is replaced by $m_g = \rho_g M_g$, but remember the change in meanings of all variables discussed earlier.

What corresponds to BB-04X sales, $s_g(t)$, in NB-87c?. Since BB-04X sales, $s_g(t)$, is new users of P_g, in NB-87c, $s\_s_g(t)$ is sales due to new users of P_g. Appendix C defines all symbols for both models.

5. Empirical

Using BB-04X, we estimate (6) with two generations of wireless telephone subscribers (ITU 2002) and four generations of IBM computer SIU (Phister 1979). We estimate (32) with two generations of copier sales (ITI 2002). Using DRAM data (Gartner 2001) for nine generations we estimate both BB-04X, (32), and NB-87c, (1). We calculate quantities such as sales, switchers and leapfrogging. We compare the same and generational p-q models. We demonstrate forecasting with BB-04X-1 by using the partially complete first and second generations to forecast the last two. We do not forecast a few points ahead as others have because such forecasts depend on no new generation during the time forecasted. In Table 2 the columns starting with the second are: generations, the model, the dependent variable, the equation number, the number of M_g (m_g in the case of NB-87c) parameters, the number of p_g and q_g parameters, the number of parameters that are significant (1-tailed test at .05 level) in the fit, overall R^2 and C as in (29).

Table 2. Empirical Summary

Data	G	Model		Eq.	Parameter Count			Overall R^2	Cannibal-ization
					M or m	p,q	Significant		
U.S. Cellular Telephone Subscribers	2	BB-04X-g	$X_g(t)$	(6)	2	4	6	0.9962	15.6%
U.S. Cellular Telephone Subscribers Differenced	2	BB-04X-g	$x_g(t)$	(10)	2	4	5	0.9599	9.5%
U.S. Cellular Telephone Subscribers	2	BB-04X-1q	$X_g(t)$	(6)	2	3	5	0.9940	5.6%
U.S. Cellular Telephone Subscribers	2	BB-04X-1	$X_g(t)$	(6)	2	2	4	0.9513	2.4%
U.S. IBM General Purpose Computers SIU	4	BB-04X-g	$X_g(t)$	(6)	4	8	12	0.9900	7.6%
U.S. IBM General Purpose Computers SIU	4	BB-04X-1	$X_g(t)$	(6)	4	2	6	0.9885	7.4%
U.S. Copier Sales	2	BB-04X-g	$s_g(t)$	(32)	2	4	6	0.9989	13.4%
U.S. Copier Sales	2	BB-04X-1	$s_g(t)$	(32)	2	2	4	0.9794	30.5%
Worldwide DRAM Chip Sales	9	NB-87-g	$s\_X_g(t)$	(1)	9	18	18	0.9830	22.2%
Worldwide DRAM Chip Sales	9	BB-04X-g	$s_g(t)$	(32)	9	18	16	0.9827	35.4%
Worldwide DRAM Chip Sales	9	NB-87-1	$s\_X_g(t)$	(1)	9	2	9	0.9118	25.6%
Worldwide DRAM Chip Sales	9	BB-04X-1	$s_g(t)$	(32)	9	2	9	0.9267	28.4%
Worldwide DRAM Bit Sales	9	NB-87-g	$s\_X_g(t)$	(1)	9	18	15	0.9986	84.8%
Worldwide DRAM Bit Sales	9	BB-04X-g	$s_g(t)$	(32)	9	8	13	0.9974	49.5%

All models were implemented in Microsoft Excel Visual Basic (VB) and SAS Proc Model. Nonlinear regression estimates were done with Bass Modeler xlnl (Bass and Isaacson 2003), Excel Solver and SAS. Table 2 summarizes the results. Appendix E presents the issues in selecting and preparing data for BB-04X and NB-87c parameter estimation. Appendix F discusses the estimation methodology. Appendix G provides the detail of the empirical analyses including data (except one set). SAS Proc Model code and Excel VB code are available from the authors. The next section presents the IBM SIU case for which detailed tabular results and charts are presented in Appendix H with interpretations.

The DRAM cases are particularly interesting for several reasons. First, prior to Bass and Bass (2001), four generations fitted was the most that had been reported in any scholarly paper. Second, we fit both NB-87c sales and BB-04X sales (see Figure 1), which resulted in about the same R^2. NB-87c views sales as a continuously purchased good while BB-04X views sales as a product purchased once each generation by each user except for leapfrogging. Both views are valid; so we are not surprised that both models achieved excellent fits. BB-04X has the advantage that the estimated M_g is the potential market, not the composite quantity $m_g = \rho_g M_g$. Even over nine generations the one p-q reduced models NB-87c-1 and BB-04X-1 had R^2's of .91 and .93, respectively, making future generation forecasts likely to be good. Another interesting aspect is that because a bit is a more appropriate unit of demand (see Appendix G) we fit bit sales by generation in addition to chip sales.

5.1. Early IBM Computer Generations

The data for the first four generations of IBM general-purpose computers installed in the USA are from Phister (1979), which as described in Bass (2002) was incompletely quoted in Mahajan and Muller (1996). We right truncated the data to 1974 because data for all computer models are not available. We changed the second generation start year to 1960 because the data showed only three SIU at the end of 1959 compared to 888 at the end of 1960. The data, tabular results and charts are in Appendices G and H.

Figure 4 compares the fits of BB-04X-g and BB-04X-1. The fits are both excellent and about the same with overall R^2's about .99 and generational R^2's between .95 and .99. The summary statistics in

Figure 5 (left) can be used to judge the effect of leapfrogging. Figure 5 (right) shows leapfroggers by generation. These numbers are plausible; however, no data are available.

Figure 4. Comparison of BB-04X-g and BB-04X-1 (6) IBM Systems in Use (SIU)

BB-04X-g Users

Estimate	g = 1	g = 2	g = 3	g = 4
M	2,602	15,503	9,912	15,502
p	0.0200	0.0329	0.0640	0.0376
q	1.2449	0.6872	0.5907	0.7166
Std Err	g = 1	g = 2	g = 3	g = 4
M	134	1,240	2,479	7,726
p	0.0113	0.0090	0.0145	0.0186
q	0.2528	0.0801	0.0789	0.1600
t ratio	g = 1	g = 2	g = 3	g = 4
M	19.48	12.51	4	2.01
p	1.77	3.64	4.4	2.02
q	4.93	8.58	7.49	4.48
	R^2 g = 1	R^2 g = 2	R^2 g = 3	R^2 g = 4
	0.9774	0.9592	0.9835	0.9843
				R^2 Overall
				0.9900
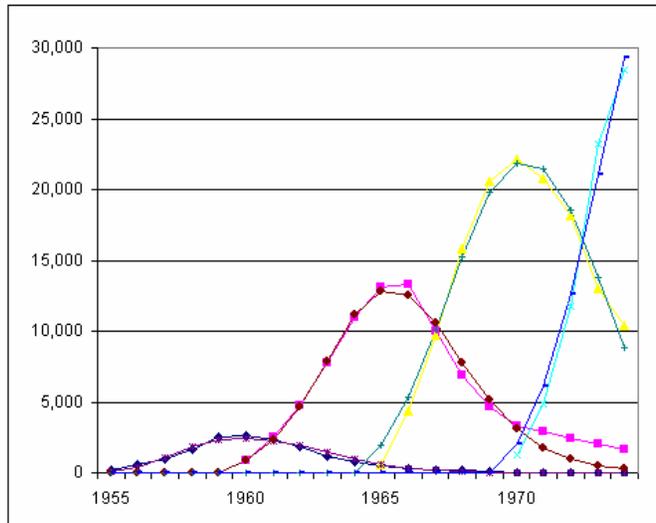
BB-04X-1 Users

Estimate	g = 1	g = 2	g = 3	g = 4
M	3,179	13,116	12,744	12,853
p	0.0455			
q	0.6737			
Std Err	g = 1	g = 2	g = 3	g = 4
M	90	596	822	2,400
p	0.0049			
q	0.0414			
t ratio	g = 1	g = 2	g = 3	g = 4
M	35.35	22	15.51	5.36
p	9.28			
q	16.26			
	R^2 g = 1	R^2 g = 2	R^2 g = 3	R^2 g = 4
	0.9756	0.9487	0.9845	0.9806
				R^2 Overall
				0.9885
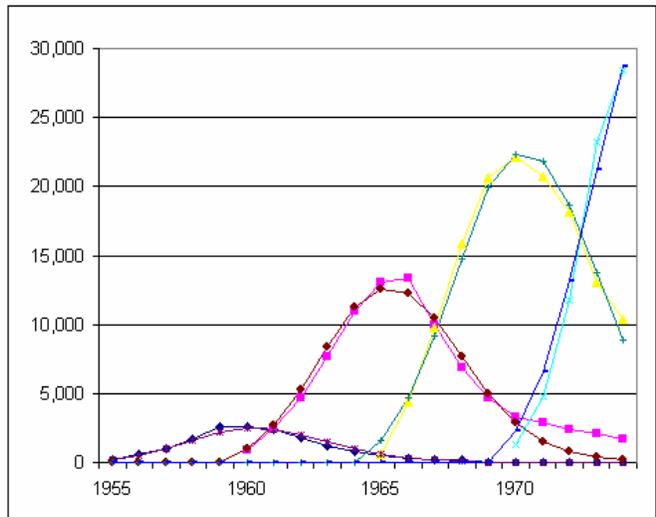
Figure 5. IBM Computer SIU Summary Statistics (Left) and Cannibalization by Generation (Right)

BB-04X-g

Leapfrogging adoptions as % of adoptions	8.3%
Leapfrogging switchers as % of switchers	7.6%
Leapfrogging as % of Potential Sales (except last gen) (Overall Cannibalization Factor)	8.2%
Adoption as % of sales	49.1%
Switching as % of sales	50.9%

Cannibalized Sales from P_g

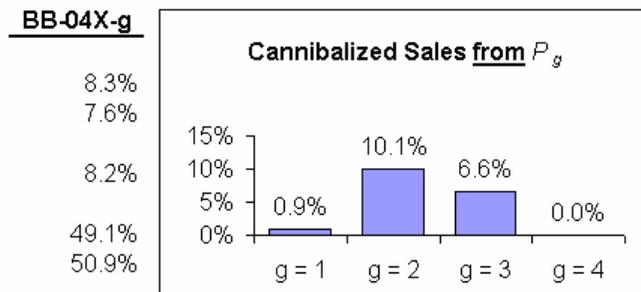g = 1: 0.9%, g = 2: 10.1%, g = 3: 6.6%, g = 4: 0.0%

Figure 6 (left) illustrates the effect of leapfrogging. We showed in (39) that M_g would have been the number of adopters of P_g had it not been for leapfrogging; however, with leapfrogging N_g is the number of adopters of P_g. Figure 6 (right) compares always positive sales to sometimes negative delta users.

Figure 6. Comparison of BB-04X-g M and N (Left)
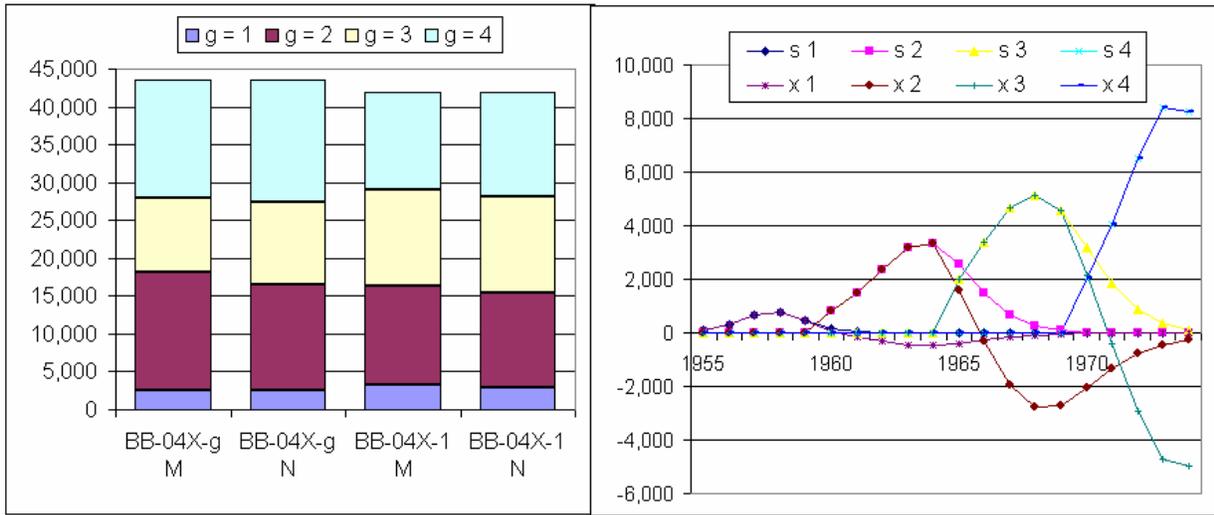and BB-04X-g Sales (32) versus Change in Users (10) (Right)

Figure 7 shows the three ways of counting users: by g of P_g being used (left), by g of Originating M_g (middle) and by g of first purchase (right). The top line of each is the same because it is the total users, which must be the same no matter how they are counted. Only users (left) declines as a new generation comes in. The middle and right graphs are similar, but net leapfrogging is the difference between them.

Figure 7. BB-04X-g:
Users by g of P_g being Used (Left), by g of Originating M_g (Middle) and by g of First Purchase (Right)
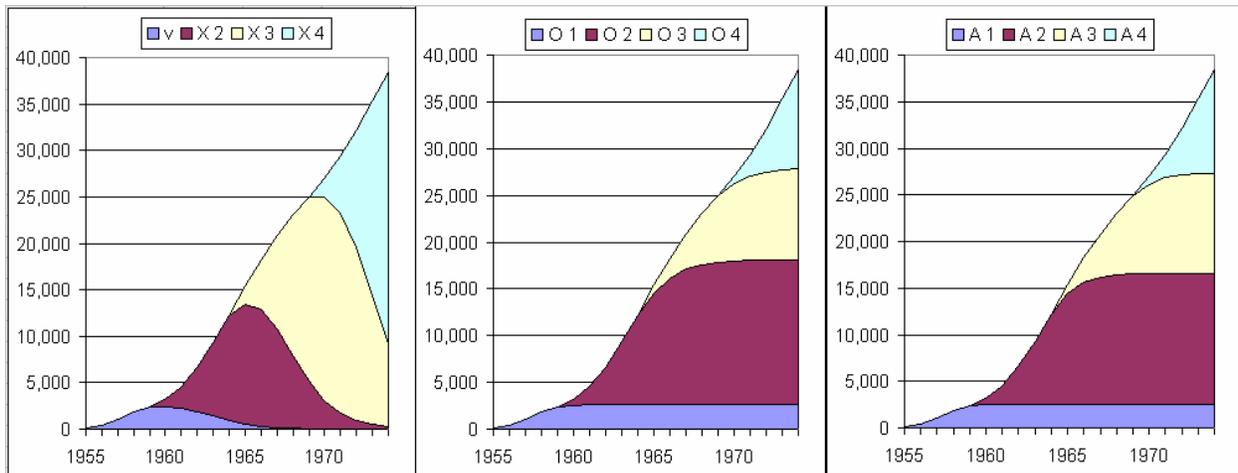
Figure 8. Diffusion of the Base Technology (Upper),
Generational Sales (Lower Left) and Adoption Sales and Replacement Sales (Lower Right)
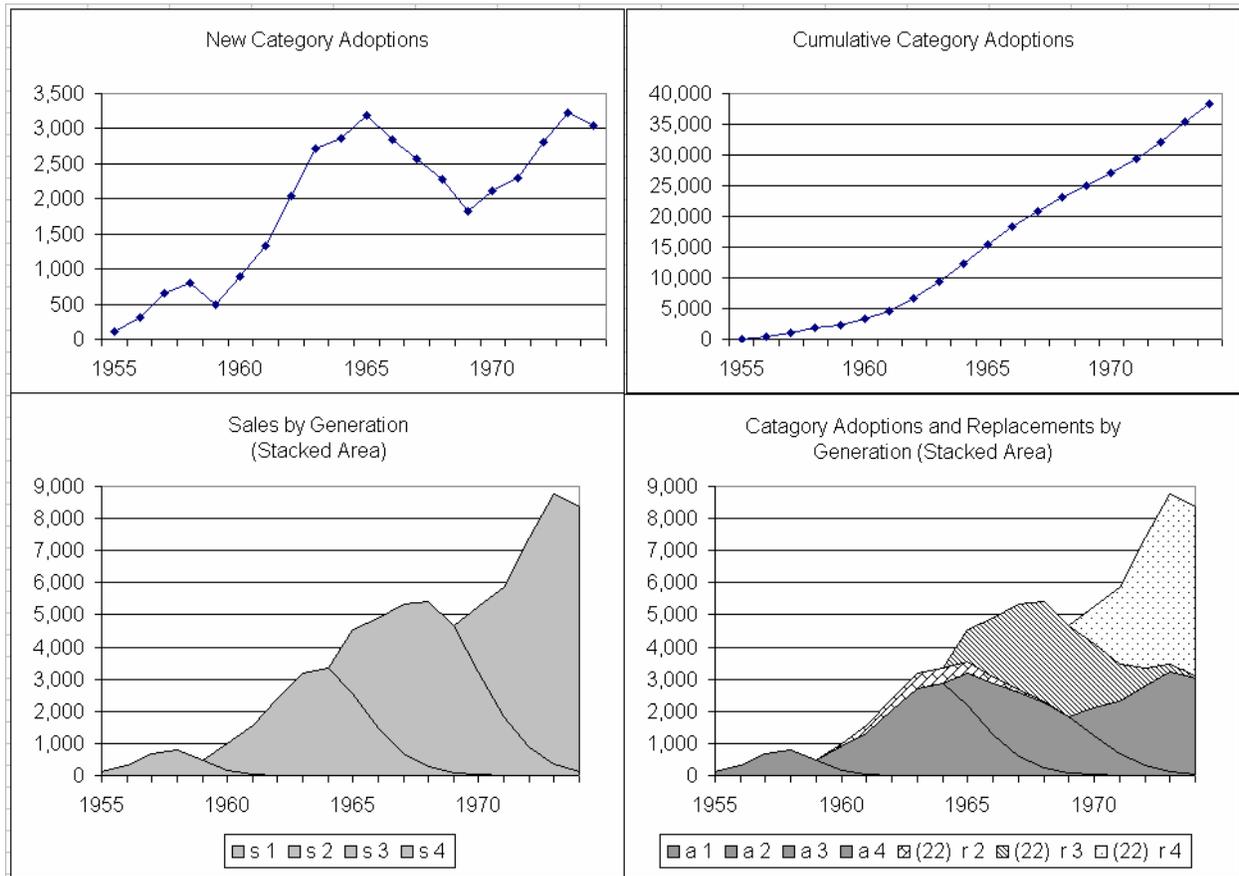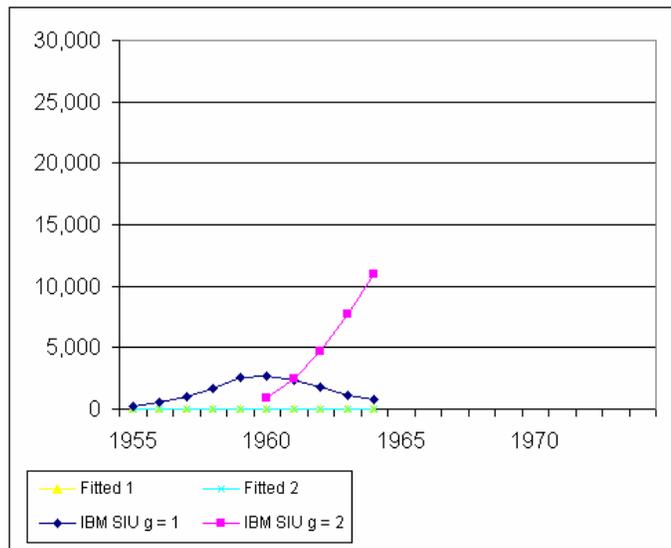

Figure 8 shows the diffusion of the base technology as new category adopters (left) and cumulative category adopters (right). These charts are as expected with surges in new customers with the introduction of new generations. Figure 8 (lower) is a stacked-area chart of sales by generation (left) and these split into adoption and replacement purchases by generation (right). Notice that the line forming the upper boundary in the two lower charts is identical because it represents total sales over all generations.

We explore the forecasting capability of BB-04X-1 by using the parameter estimates for two generations through 1964 as shown in Figure 9 (upper) to forecast the rest of the second generation and the entire third and fourth generations. We expect this to be feasible because the generations are regular and frequent (every five years). The forecast requires only information that would be available in 1964 including the assumption that generations will occur every five years and common marketing research data. Figure 9 (lower) is the forecast. We increase M_2 slightly to allow for some leapfrogging when we add
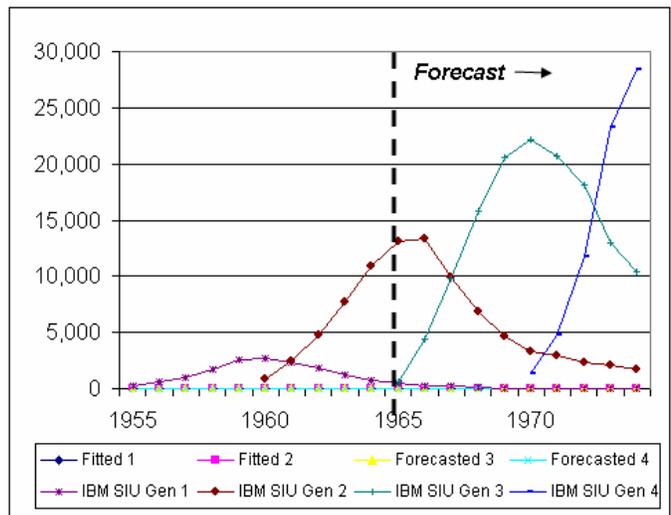
two more generations. We must guesstimate M_3 and M_4. We used the same number for the incremental market potential as generation 2; however, that would not normally be the case. Normally we would use marketing research techniques typically used to obtain market sizing, customer purchase intentions and customer replacement patterns. Generation four incremental market potential can also be guesstimated; however, we would expect it to be fuzzier. The R^2's in Figure 9 (lower) are measures of the forecast relative to the historical data. Although this forecasting method depends on accurate market sizing, it does not depend on guesstimating p_g and q_g as using BB-04X-g would.

Figure 9. Fit of Two-Generations to 1965 as First Step (Upper)
and Forecast Using p-q Parameters from First Step Compared to Actual (Lower)

6.   Summary and Conclusions

We have defined the required components of a complete generational diffusion model as a necessary pre-liminary to completing the two models that have heretofore been known as the Norton-Bass Model. We have for the first time differentiated the classic Norton-Bass from its Embedded Model, which is more widely applicable since it is a model of users (e.g., SIU, installed base, subscribers, penetration) and of sales of a product purchased once each generation by each user except for leapfrogging. We have derived for the Embedded Model 39 new quantities, for example: leapfroggers, switchers, sales, replacements and renewals. For the classic Norton-Bass we have derived sales due to each of the newly identified quanti-ties (e.g., leapfroggers) in the Embedded Model. We have estimated the Embedded Model by fitting it to sales, SIU or subscriber data for several IT categories including: cellular telephone, IBM computer sys-tems, copiers, DRAM chips and DRAM bits and have demonstrated the newly derived quantities. We have shown that both NB-87c sales-c and BB-04X sales-1 are good fits to nine generations of DRAM, which is not surprising since both views of the phenomenon are valid.

With regard to the "inherent inconsistency" when deriving the first-time adoption model for multiple generations observed by Danaher et al. (2001), we have shown that there is none by doing the derivation.

With regard to the p-q issue we conclude that both variants are useful. NB-87c-g and BB-04X-g are good matches to their respective phenomena as demonstrated by our empirical analyses. For forecasting, however, NB-87c-g and BB-04X-g are useful only for extending a generation a few time periods and then only if a new generation has not entered. For forecasting entire generations prior to or after only a few data points, NB-87c-g and BB-04X-g require guesstimating three parameters per generation. But, you say, can't prior generations be used as analogies? Yes, that is effectively what using NB-87c-1 or BB-04X-1 does: it determines one set of parameters p and q that give a good overall fit. If the market has changed considerably over time, then more than one p-q set can be used. We expect that product catego-ries with frequent, regular generations are better candidates for forecasting one or more generations ahead than are categories with irregular and infrequent generations.

With regard to the leapfrogging issue we have demonstrated that the classic Norton-Bass and the Embedded Model have leapfroggers by deriving the equations for them and estimating them empirically.

7.   Areas for Future Research

Our review of the generational diffusion model literature and our exploration of the newly defined model have raised the following questions: (1) Need all significant substitutable products and successive products be included in the data used to estimate generational models? (2) How should the forecasting capability of generational models be judged? (3) Is leapfrogging -- a decomposed quantity with no empirical validation -- a desirable property of a generational model? (4) What is the effect of left truncating the data? (5) What is the effect of omitting the data of a generation that overlapped with others during the modeled timeframe? We give our opinions on some of these issues in Appendices E and F; but more research is needed on all of them. Other possibilities for this research stream include (1) comparing the new model BB-04X both theoretically and empirically to the models MM-96, DHP-01, and BB-01; (2) exploring leapfrogging with empirical evidence, qualitative information and simulation; and (3) using BB-04X to explore pricing, generational timing and competition.

Although we have derived 80 BB-04X equations for quantities that are common to all generational phenomena that can be measured as counts of users, there are still other equations that can be derived that have managerial importance; for example, to name a few (all by generation): expected time from a purchase to its replacement, mix of prior product ownership, percentage of switchers that leapfrog to the last generation, time to the peak, time to the peak in category adoptions, time to saturation of the total market and the set of all possible purchase histories as well as the number of users having each history.

A third model is implied by this paper. NB-87c and BB-04X can be used together by first estimating or specifying the p_g, q_g and M_g for BB-04X then estimating or specifying ρ_g, for NB-87c while using the other parameters from BB-04X. Example uses include cellular subscribers with minutes of usage and computer-printer users with cartridge sales. We will leave this exploration for another time.

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